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Geometrical phase in the cyclic evolution of non-Hermitian systems

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Abstract. We derive, by a biorthonormal state approach, the analogy of Berry's phase factor for open, non-conservative systems, for both adiabatic and non-adiabatic evolution. In the latter case, a (non-unitary) evolution operator method is exploited. An application is given to the optical supermode propagation in the free-electron laser.

1. Introduction

The subject of the topological phase factor arising in the dynamical evolution of quantum systems (as first discovered and investigated by Berry [1]) has attracted in the past few years a considerable amount of interest, from both the theoretical [2-8] and the experimental viewpoints [9]. Theoretical developments include, for example, the elucidation of the geometrical meaning of Berry's phase [2-6] and a number of applications to quantum mechanics and quantum field theory [7] (including gauge theories and generalized coherent and squeezed states [8]).

Although Berry's phase factor arises in the evolution of a system interacting with a surrounding, virtually all of the existing literature has been concerned with closed systems, driven by Hermitian Hamiltonians. It is only recently [10, 11] that Berry's phase has been considered for open, dissipative systems. In [11], such a problem is approached in a density-matrix framework, by a superoperator formalism. We want here to derive the generalization of Berry's phase for systems with non-Hermitian (nH) dynamic evolution by a biorthonormal-state method, which revealed itself very fruitful in the treatment of a variety of physical problems (ranging from multiphoton ionization [12-14] to transverse mode propagation [15] to free-electron laser theory [16]).

The paper is organized as follows. In section 2 we briefly review the biorthonormal state formalism for nH Hamiltonians, and derive a generalization of Berry's phase for nH systems in the adiabatic approximation. In section 3 we exploit an evolution operator method [17] (suitably generalized to nH systems [14]) to give an alternative derivation of the nH Berry's factor which is independent of the adiabatic hypothesis. An application to the optical supermode propagation in the free-electron laser (FEL) is given in section 4. Section 5 concludes the paper.

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2. Berry's phase for non-Hermitian systems

Let us consider a system ruled by the nH Hamiltonian (nHH) (pseudo-Hamiltonian) $\hat{H}$ ($\hat{H}^+ \neq \hat{H}$). The two different sets of eigenstates of $\hat{H}$ and $\hat{H}^+$

$$\hat{H}|\varphi_n\rangle = \lambda_n|\varphi_n\rangle \tag{2.1}$$

$$\hat{H}|\chi_m\rangle = \lambda_m^*|\chi_m\rangle \tag{2.2}$$

are biorthogonal to each other (provided the complex eigenvalues λ_n are not degenerate):

$$\langle\chi_m|\varphi_n\rangle = \langle\varphi_m|\chi_n\rangle = \delta_{mn}. \tag{2.3}$$

Any system state can be expanded in terms of either the φs or the χs as follows:

$$|\Psi\rangle = \sum_m c_m|\chi_m\rangle \qquad c_m = \langle\varphi_m|\Psi\rangle \tag{2.4}$$

$$|\Psi\rangle = \sum_m \bar{c}_m|\varphi_m\rangle \qquad \bar{c}_m = \langle\chi_m|\Psi\rangle. \tag{2.5}$$

Moreover, the closure relation has the form

$$\sum_n |\chi_n\rangle\langle\varphi_n| = \sum_n |\varphi_n\rangle\langle\chi_n| = \hat{I}. \tag{2.6}$$

Suppose now that $\hat{H}$ and $\hat{H}^+$ are functions of some set of parameters $\boldsymbol{R}(t)$†, which are slowly changed. The time evolution of the system is given by the nH Schrödinger equation ($\hbar = 1$)

$$\mathrm{i}\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}(\boldsymbol{R}(t))|\Psi(t)\rangle. \tag{2.7}$$

We want to study the behaviour of the system in the time interval $[0, T]$, in the hypothesis that at the initial instant $t=0$ it is in an eigenstate of $\hat{H}$, at $t=T$ the parameters $\boldsymbol{R}$ are returned to their initial values ($\boldsymbol{R}(T)=\boldsymbol{R}(0)$), and the adiabatic theorem holds. Then, the nH system remains at any instant in an eigenstate of the Hamiltonian $\hat{H}(t)$, apart from a damping factor due to the non-Hermiticity of $\hat{H}$ (and, therefore, to the non-unitarity of the corresponding time evolution).

Thus, according to the adiabatic theorem and Berry's results, we can write for the system wavefunction at time t:

$$|\Psi(t)\rangle = \exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\,\lambda_n(\boldsymbol{R}(t'))\right)\exp(\mathrm{i}\bar{\gamma}_n(t))|\varphi_n(\boldsymbol{R}(t))\rangle \tag{2.8}$$

where $\bar{\gamma}_n(t)$ is the nH analogy of Berry's phase, whose expression we want now to find. We have, for the time derivative (herafter denoted by a dot) of $|\Psi\rangle$:

$$|\dot{\Psi}(t)\rangle = -\mathrm{i}\lambda_n|\Psi\rangle + \mathrm{i}\dot{\bar{\gamma}}_n|\Psi\rangle + \exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\,\lambda_n\right)\exp(\mathrm{i}\bar{\gamma}_n)|\nabla_{\boldsymbol{R}}\varphi_n\rangle\cdot\dot{\boldsymbol{R}} \tag{2.9}$$

(here $\nabla_{\boldsymbol{R}}$ is the gradient operator with respect to $\boldsymbol{R}$). On account of (2.1) and (2.7), we get

$$\dot{\bar{\gamma}}_n|\Psi\rangle = \mathrm{i}\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\,\lambda_n\right)\exp(-\mathrm{i}\bar{\gamma}_n)|\nabla_{\boldsymbol{R}}\varphi_n\rangle\cdot\dot{\boldsymbol{R}}. \tag{2.10}$$

† Clearly, $\boldsymbol{R}(T)$ is in general a vector in an N-dimensional Euclidean space, but for simplicity we shall consider the case $N=3$.

Due to the biorthogonality of the $\hat{H}$, $\hat{H}^+$ eigenstates, $|\Psi(t)\rangle$ can be also expressed as

$$|\Psi(t)\rangle = \exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\,\lambda_n^*\right)\exp(\mathrm{i}\bar{\gamma}_n^*(t))|\chi_n(\boldsymbol{R}(t))\rangle. \tag{2.11}$$

By taking the scalar product of both sides of (2.10) by (2.11), we find

$$\langle\Psi|\dot{\bar{\gamma}}_n|\Psi\rangle = \mathrm{i}\langle\chi_n|\boldsymbol{\nabla}_{\boldsymbol{R}}\varphi_n\rangle\cdot\dot{\boldsymbol{R}} \tag{2.12}$$

and therefore

$$\dot{\bar{\gamma}}_n(t) = \mathrm{i}\langle\chi_n(\boldsymbol{R}(t))|\boldsymbol{\nabla}_{\boldsymbol{R}}\varphi_n(\boldsymbol{R}(t))\rangle\cdot\dot{\boldsymbol{R}}. \tag{2.13}$$

The system excursion between the times $t=0$ and $t=T$ can be pictured, in parameter space, as transport round a closed path C. The total change of $|\Psi\rangle$ round C is therefore given by

$$|\Psi(T)\rangle = \exp(\mathrm{i}\bar{\gamma}_n(C))\exp\left(-\mathrm{i}\int_0^T \mathrm{d}t\,\lambda_n(\boldsymbol{R}(t))\right)|\Psi(0)\rangle \tag{2.14}$$

where $\bar{\gamma}_n(C)$ is the generalization of Berry's geometrical phase to nH systems, and reads

$$\bar{\gamma}_n(C) = \mathrm{i}\oint_C \bar{\boldsymbol{A}}\cdot\mathrm{d}\boldsymbol{R} \equiv \mathrm{i}\oint_C \langle\chi_n(\boldsymbol{R}(t))|\boldsymbol{\nabla}_{\boldsymbol{R}}\varphi_n(\boldsymbol{R}(t))\rangle\cdot\mathrm{d}\boldsymbol{R} \tag{2.15}$$

where $\bar{\boldsymbol{A}} = \langle\chi_n|\boldsymbol{\nabla}_{\boldsymbol{R}}\varphi_n\rangle$ is the nH connection (or pseudopotential).

A few comments are in order. First of all, let us notice explicitly that, due to the non-Hermiticity of $\hat{H}$, $\bar{\gamma}_n(C)$ is no longer real. Let us recall that, in the Hermitian case, the reality of the standard Berry's phase γ_n is connected to the normalization of the Hamiltonian eigenstates. In the present case, the normalization condition is replaced by the binormalization relation (2.3), thus allowing for a non-real $\bar{\gamma}_n$. Then, for nH systems, the transport around C induces a change in the wavefunction, which no longer amounts to a mere phase factor. It is easily seen that this result (which agrees with the findings in [11]) is exactly the geometrical analogy of the modification in the dynamic factor in passing from Hermitian to non-Hermitian evolution:

$$\begin{aligned}&\exp\left(-\mathrm{i}\int_0^t E_m(t')\,\mathrm{d}t'\right)\to\exp\left(-\mathrm{i}\int_0^t \lambda_m(t')\,\mathrm{d}t'\right)\\ &\exp(\mathrm{i}\gamma_n(C))\to\exp(\mathrm{i}\bar{\gamma}_n(C))\end{aligned} \tag{2.16}$$

(E_n, γ_n real; λ_n, $\bar{\gamma}_n$ complex). As a consequence, we get two damping factors, one of dynamic and one of geometrical origin.

It is easy to realize that the direct evaluation of $|\boldsymbol{\nabla}_{\boldsymbol{R}}\varphi_n\rangle$ in (2.15) can, in some cases, present the same difficulties as for the standard Berry phase. They can be avoided in exactly the same way, i.e. transforming the line integral (2.15) into a surface integral by the Stokes theorem. One has

$$\begin{aligned}\bar{\gamma}_n(C) &= -\mathrm{i}\iint_S \mathrm{d}\boldsymbol{s}\cdot\boldsymbol{\nabla}_{\boldsymbol{R}}\times\langle\chi_n|\boldsymbol{\nabla}_{\boldsymbol{R}}\varphi_n\rangle\\ &= -\mathrm{i}\iint_S \mathrm{d}\boldsymbol{s}\cdot\langle\boldsymbol{\nabla}_{\boldsymbol{R}}\chi_n|\times|\boldsymbol{\nabla}_{\boldsymbol{R}}\varphi_n\rangle\\ &= -\mathrm{i}\iint_S \mathrm{d}\boldsymbol{s}\cdot\sum_{m\neq n}\langle\boldsymbol{\nabla}_{\boldsymbol{R}}\chi_n|\varphi_n\rangle\times\langle\chi_m|\boldsymbol{\nabla}_{\boldsymbol{R}}\varphi_n\rangle\end{aligned} \tag{2.17}$$

where in the last step we inserted the unity decomposition (2.6). The exclusion of n in the summation is justified by the fact that, due to the binormalization relation (2.3), the vectors $\langle\nabla\chi_n|\varphi_n\rangle$ and $\langle\chi_n|\nabla\varphi_n\rangle$ are antiparallel†.

Alternative, useful expressions of the off-diagonal elements are obtained by differentiating (2.1) and (2.2). We get, for instance,

$$\nabla(\hat{H})|\varphi_n\rangle) = (\nabla\hat{H})|\varphi_n\rangle + \hat{H}|\nabla\varphi_n\rangle = \lambda_n|\nabla\varphi_n\rangle + (\nabla\lambda_n)|\varphi_n\rangle. \tag{2.18}$$

Multiplying both sides of (2.18) by $\langle\chi_m|$, we easily find that

$$\langle\chi_m|\nabla\varphi_n\rangle = \frac{\langle\chi_m|(\nabla\hat{H})|\varphi_n\rangle}{\lambda_n - \lambda_m} \qquad n \neq m. \tag{2.19}$$

Analogously, from (2.2),

$$\langle\varphi_m|\nabla\chi_n\rangle = \frac{\langle\varphi_m|(\nabla\hat{H}^+)|\chi_n\rangle}{\lambda_m^* - \lambda_n^*} \qquad m \neq n. \tag{2.20}$$

Thus, $\bar{\gamma}_n(C)$ can be written as

$$\bar{\gamma}_n(C) = -\iint_S \mathrm{d}s \cdot \bar{V}_n(\boldsymbol{R}) \tag{2.21}$$

where

$$\bar{V}_n(\boldsymbol{R}) = \mathrm{i} \sum_{m \neq n} \frac{\langle\chi_n|\nabla_{\boldsymbol{R}}\hat{H}|\varphi_m\rangle \times \langle\chi_m|\nabla_{\boldsymbol{R}}\hat{H}|\varphi_n\rangle}{(\lambda_n - \lambda_m)^2}. \tag{2.22}$$

3. The evolution operator method

In the previous derivation of Berry's phase for nH systems, we have assumed the evolution to occur adiabatically. However, as first proved by Aharonov and Anandan [4], the adiabatic hypothesis is by no means necessary in order that a Hermitian system develops a topological factor. We want now to show that this also holds true in the nH case, by exploiting an evolution operator method [17], suitably extended to nH systems [14].

Let us denote by $\hat{U}(t)$ the evolution operator associated with the nH Schrödinger equation (2.7): we thus have

$$|\Psi(t)\rangle = \hat{U}(t)|\Psi(0)\rangle. \tag{3.1}$$

The equation obeyed by $\hat{U}(t)$ is

$$\mathrm{i}\frac{\partial}{\partial t}\hat{U}(t) = \hat{H}\hat{U} \qquad \hat{U}(0) = \hat{I}. \tag{3.2}$$

Of course, $\hat{U}$ is not unitary, and therefore $\hat{U}\hat{U}^+ \neq \hat{I}$. Let us introduce the (non-unitary) evolution operator $\hat{\hat{U}}(t)$ associated to $\hat{H}^+$. It can be shown [14] that

$$\hat{U}\hat{\hat{U}}^+ = \hat{\hat{U}}^+\hat{U} = \hat{I} \tag{3.3}$$

† Indeed

$$\langle\chi_n|\varphi_n\rangle = 1 \Rightarrow \langle\nabla\chi_n|\varphi_n\rangle + \langle\chi_n|\nabla\varphi_n\rangle = 0.$$

which implies that the states

$$|\varphi_n(t)\rangle = \hat{U}(t)|\varphi_n(0)\rangle \qquad |\chi_n(t)\rangle = \hat{\hat{U}}(t)|\chi_n(0)\rangle \tag{3.4}$$

are biorthogonal states at any instant (notice, however, that, in general, they are no longer instantaneous eigenstates of $\hat{H}$).

Let us define the matrix elements of any nH operator $\hat{A}$ with respect to the biorthonormal states $|\chi_m\rangle$, $|\varphi_n\rangle$ as

$$A_{mn} = \langle\chi_m|\hat{A}|\varphi_n\rangle. \tag{3.5}$$

Then, it is easily seen that $\hat{A}$ can be diagonalized with respect to the biorthogonal set by a biunitary transformation [13, 14]:

$$\hat{A}' = \hat{\hat{W}}^{+}\hat{A}\hat{W} \tag{3.6}$$

(see, for example [13] for the explicit form of $\hat{W}$, $\hat{\hat{W}}$). Instead, the adjoint operator $\hat{A}^{+}$ is diagonalized by the transformation

$$\hat{A}^{+\prime} = \hat{W}^{+}\hat{A}^{+}\hat{\hat{W}}. \tag{3.7}$$

The converse is obviously true if the matrix elements of $\hat{A}$ are defined as

$$\bar{A}_{mn} = \langle\varphi_m|\hat{A}|\chi_n\rangle. \tag{3.8}$$

We assume now that both the evolution operators $\hat{U}$ and $\hat{\hat{U}}$ can be written as products of two non-unitary operators†, i.e. make the ansatz

$$\hat{U}(t) = \hat{S}(t)\hat{R}(t) \tag{3.9}$$

$$\hat{\hat{U}}(t) = \hat{\hat{S}}(t)\hat{\hat{R}}(t) \tag{3.10}$$

($\hat{S}^{+} \neq \hat{S}^{-1}$; $\hat{R}^{+} \neq \hat{R}^{-1}$; $\hat{\hat{S}}^{+} \neq \hat{\hat{S}}^{-1}$; $\hat{\hat{R}}^{+} \neq \hat{\hat{R}}^{-1}$). From (3.3), it follows that

$$\hat{\hat{S}}^{+}\hat{S} = \hat{S}\hat{\hat{S}}^{+} = \hat{I} \qquad \hat{\hat{R}}^{+}\hat{R} = \hat{R}\hat{\hat{R}}^{+} = \hat{I}. \tag{3.11}$$

Inserting (3.9) in (3.2), and using (3.10) and (3.11), we easily get

$$\hat{\hat{S}}^{+}\left[\hat{H} - \mathrm{i}\frac{\partial}{\partial t}\right]\hat{S} = \mathrm{i}\hat{R}\hat{\hat{R}}^{+}. \tag{3.12}$$

Let us put

$$\hat{\mathscr{H}}(t) = \mathrm{i}\hat{R}\hat{\hat{R}}^{+} \tag{3.13}$$

which obviously is not Hermitian.

It is easy to see that the non-unitary transformation induced by $\hat{S}(t)$, i.e.

$$|\Psi(t)\rangle = \hat{S}(t)|\Phi(t)\rangle \tag{3.14}$$

leads from (2.7) to the new nH evolution equation

$$\mathrm{i}\frac{\partial}{\partial t}|\Phi(t)\rangle = \hat{\mathscr{H}}(t)|\Phi(t)\rangle \tag{3.15}$$

† Clearly, this can be done in infinitely many ways.

in which the operator $\hat{\mathscr{H}}(t)$ defined by (3.13) plays the role of pseudo-Hamiltonian. The formal solution of (3.15) reads†

$$|\Phi(t)\rangle = \exp\left(-\mathrm{i}\int_0^t \hat{\mathscr{H}}(t')\,\mathrm{d}t'\right)|\Phi(0)\rangle \tag{3.16}$$

and therefore, in the original representation,

$$|\Psi(t)\rangle = \hat{S}(t)\exp\left(-\mathrm{i}\int_0^t \hat{\mathscr{H}}(t')\,\mathrm{d}t'\right)|\Psi(0)\rangle. \tag{3.17}$$

To explicitly solve (3.16) and (3.17) we have to assume that $\hat{\mathscr{H}}$ is diagonalizable in the biorthonormal basis $|\chi_n(0)\rangle$, $|\varphi_n(0)\rangle$, according to (3.5) and (3.6). This is clearly true if and only if $\hat{R}$ and $\hat{\bar{R}}^+$ are diagonal in the same basis. By putting

$$R_{mn}(t) = \delta_{mn}\exp(\theta_n(t)) \tag{3.18}$$

with θ_n complex, we get, on account of (3.11),

$$\mathscr{H}_{mn} = \mathrm{i}\delta_{mn}\dot{\theta}_n. \tag{3.19}$$

For an initial state $|\Psi(0)\rangle = |\varphi_n(0)\rangle$, we have, from (3.12),

$$\left(\hat{\bar{S}}^+\left[\hat{H} - \mathrm{i}\frac{\partial}{\partial t}\right]\hat{S}\right)|\varphi_n(0)\rangle = \hat{\mathscr{H}}|\varphi_n(0)\rangle = \mathrm{i}\dot{\theta}_n|\varphi_n(0)\rangle \tag{3.20}$$

and therefore

$$\dot{\theta}_n = -\mathrm{i}\langle\chi_n(0)|\hat{\bar{S}}^+\hat{H}\hat{S}|\varphi_n(0)\rangle - \langle\chi_n(0)|\hat{\bar{S}}^+\dot{\hat{S}}|\varphi_n(0)\rangle. \tag{3.21}$$

After integration, we find the expression of the complex phase θ_n:

$$\theta_n(t) = -\mathrm{i}\int_0^t \langle\chi_n(t')|\hat{H}|\varphi_n(t')\rangle\,\mathrm{d}t' - \int_0^t \langle\chi_n(t')|\dot{\varphi}_n(t')\rangle\,\mathrm{d}t' \tag{3.22}$$

where, hereafter, the time-dependent states are obtained by the action of the operators $\hat{S}$, $\hat{\bar{S}}$. Then, (3.17) becomes

$$|\Psi(t)\rangle = \exp\{-\mathrm{i}\bar{\gamma}_n^{\mathrm{D}}(t)\}\exp\{\mathrm{i}\bar{\gamma}_n(t)\}|\varphi_n(t)\rangle \tag{3.23}$$

where $\bar{\gamma}_n^{\mathrm{D}}$ and $\bar{\gamma}_n$ are the nH dynamic and topological phases, given by

$$\bar{\gamma}_n^{\mathrm{D}}(t) = \int_0^t \langle\chi_n(t')|\hat{H}|\varphi_n(t')\rangle\,\mathrm{d}t' \tag{3.24}$$

$$\bar{\gamma}_n(t) = \mathrm{i}\int_0^t \langle\chi_n(t')|\dot{\varphi}_n(t')\rangle\,\mathrm{d}t'. \tag{3.25}$$

Notice that in the derivation of (3.24) and (3.25) we made no recourse to the adiabatic hypothesis; in other words $|\varphi_n(t)\rangle$ is not, in general, an instantaneous eigenstate of $\hat{H}$. Clearly, in the assumption of an adiabatic evolution of the system, for $\hat{H} = \hat{H}(\boldsymbol{R}(t))$ and for a closed path in parameter space, we recover the expressions (2.8) and (2.15) of the nH Berry phase.

† However, let us stress that the actual evaluation of the exponential operator in (3.16) and (3.17) involves problems of time ordering (because the operators $\hat{\mathscr{H}}(t)$ do not commute at different times).

Let us stress that the importance of the evolution operator approach developed in this section lies not only in its independence of the adiabatic hypothesis, but also in the fact that one can take advantage of well-stated techniques of operator ordering [18] (like the Wei–Norman method) (suitably extended to nH systems [14]).

4. Non-Hermitian Berry's phase in the free-electron laser supermode propagation

We want now to apply the results of the previous sections to the optical pulse propagation in free-electron laser (FEL).

For the reader's convenience, let us recall the main FEL physics [19]. The FEL is a coherent source of radiation in which the active medium consists of an ultrarelativistic electron beam moving in a magnetic undulator with N periods and wavelength λ_u. The emitted radiation is stored in an optical cavity and reinforced by a new copropagating electron beam. In FELs driven by radio-frequency (RF) accelerators, the electron beam has a structure characterized by a series of microbunches (of longitudinal length σ_z) with a distance fixed by the RF period. The bunched structure of the electron beam induces an analogous structure in the optical field.

Therefore, to have self-sustained laser action, the electron and optical bunches must be synchronized in such a way that after one round trip the laser bunch overlaps a freshly injected electron bunch.

A typical configuration for an RF operating FEL is shown in figure 1. Due to the different speeds of the electron and laser bunches, the FEL exhibits the so-called 'lethargic' behaviour, i.e. the front side of the optical pulse experiences less gain than the backward part, so that the centroid of the laser pulse is slowed down. This lethargy effect affects the synchronism condition: to have timing between the electrons and optical field one must reduce the length of the cavity by a quantity δL to compensate for the velocity reduction of the laser pulse.

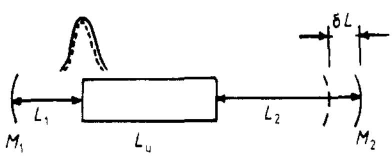

Figure 1. Typical FEL configuration: full line, electron bunch; broken line, laser bunch; M_1, fixed mirror; M_2, movable mirror; $L_c = L_1 + L_u + L_2$, total cavity length; δL, cavity mismatch.

The propagation of the optical field in a RF-operating FEL can be accounted for by means of an expansion in longitudinal modes of the optical cavity. Because of the very large numbers of interacting modes (typically a few thousands), it is practically impossible to follow the single mode evolution and it has been proved convenient to analyse those clusters of longitudinal modes ('supermodes') which reproduce themselves unchanged after each round trip (although their phases and amplitudes can vary) [19]. These FEL supermodes have been shown to be the eigenfunctions of an integrodifferential equation, which can be reduced to an nH Schrödinger-type equation [16] in the hypothesis (verified in most of the experimental cases) that the slippage ($\Delta = N\lambda$, λ being the resonant wavelength) is small compared to the RMS electron bunch length σ_z. Indeed, in this case the FEL pulse propagation in the forward z direction can be cast in the form [16]

$$\frac{\partial}{\partial \tau} E_F(Z, \tau) = \hat{H} E_F(Z, \tau) \tag{4.1}$$

where E_F is the forward propagating, slowly varying part of the optical electric field, Z is essentially the longitudinal coordinate, the dimensionless time τ is a discrete time related to the number of round trips, and $\hat{H}$ is the nH Hamiltonian,

$$\hat{H} = \Omega_1 \hat{K}_+ + \Omega_2 \hat{K}_- + \Omega_3 \hat{a} + \Omega_4 \hat{I} \tag{4.2}$$

with†

$$\hat{a} = \frac{\partial}{\partial Z} \qquad \hat{K}_+ = \tfrac{1}{2} Z^2 \qquad \hat{K}_- = \frac{1}{2} \frac{\partial^2}{\partial Z^2}. \tag{4.3}$$

Moreover, the Ω_is ($i = 1, 2, 3, 4$) are complex functions, depending on the FEL physical parameters (see table 1):

$$\Omega_1 = -G_1(\nu_0) \qquad \Omega_2 = \mu_c^2 G_4(\nu_0) \qquad \Omega_3 = \mu_c(G_3^{(1)} - \vartheta) \qquad \Omega_4 = G_1 - \frac{\gamma_T}{g_0} \tag{4.4}$$

The parameter ϑ is related to the cavity desynchronism necessary to compensate for the lethargic effect. Moreover, G_1 is the complex gain function

$$G_1 = -2\pi \frac{\partial}{\partial \nu_0} \left(1 + \mathrm{i} \frac{\partial}{\partial \nu_0}\right) \left(\frac{\sin(\nu_0/2)}{(\nu_0/2)} \exp(\mathrm{i}\nu_0/2)\right) \tag{4.5}$$

and G_3, G_4 are given by

$$G_3 = -\mathrm{i} \frac{\partial G_1}{\partial \nu_0} \qquad G_4 = -\frac{\partial^2}{\partial \nu_0^2} G_1. \tag{4.6}$$

The adjoint of $\hat{H}$ is obviously

$$\hat{H}^+ = \Omega_1^* \hat{K}_+ + \Omega_2^* \hat{K}_- - \Omega_3^* \hat{a} + \Omega_4^* \hat{I}. \tag{4.7}$$

Table 1. List of the symbols used for the FEL optical propagation in section 4.

ν_0	resonance parameter
σ_z	RMS longitudinal bunch length
N	number of passes
L_c	optical cavity length
λ	resonant wavelength
$\Delta = N\lambda$	slippage distance
$\mu_c = \Delta/\sigma_z$	coupling parameter
δL	cavity detuning ($L_c - \delta L$ = effective cavity length)
g_0	gain coefficient
γ_T	cavity losses
$\theta = 4\delta L/\Delta g_0$	delay parameter

† Equation (4.3) is easily seen to express the coordinate representation of the annihilation operator $\hat{a}$ and of the ladder operators of the SU(1, 1) algebra. Indeed, the RHS of (4.2) is readily recognized as an element of the semidirect sum SU(1, 1)$\oplus h(4)$($h(4)$ being the Weyl–Heisenberg group).

The eigenstates of $\hat{H}$ and $\hat{H}^+$ read, respectively,

$$\phi_n(Z)=\frac{N_n f(\vartheta)}{(n!2^n\sqrt{\pi}\,\sigma_E)^{1/2}}H_n\left(\frac{Z}{\sigma_E}\right)\exp\left(-\frac{1}{2\sigma_E^2}(Z-Z_0)^2\right) \tag{4.8}$$

$$\chi_m(Z)=\frac{\bar{N}_m\bar{f}(\vartheta)}{(m!2^m\sqrt{\pi}\,\bar{\sigma}_E)^{1/2}}H_m\left(\frac{Z}{\bar{\sigma}_E}\right)\exp\left(-\frac{1}{2\bar{\sigma}_E^2}(Z-\bar{Z}_0)^2\right) \tag{4.9}$$

where $H_n(\cdot)$ are the Hermite polynomials, N_n, $\bar{N}_m$ are normalization factors and

$$\sigma_E^2=\mu_c\sqrt{\frac{G_4}{G_1}} \qquad Z_0=\frac{G_3-\vartheta}{\sqrt{G_1G_4}}$$

$$f(\vartheta)=\exp\left(\frac{(G_3-\vartheta)^2}{2\mu_c G_4\sqrt{G_1G_4}}\right) \tag{4.10}$$

$$\bar{Z}_0=-Z_0^* \qquad \bar{\sigma}_E^2=\sigma_E^{*2} \qquad \bar{f}(\vartheta)=\exp\left(\frac{(G_3^*-\vartheta)^2}{2\mu_c G_4^*\sqrt{G_1^*G_4^*}}\right).$$

The parameter Z_0 represents the shift of the laser packet with respect to the electron bunch. The eigenvalues of $\hat{H}$ have the form

$$\lambda_n=G_1-\frac{\gamma_T}{g_0}-(n+\tfrac{1}{2})\mu_c\sqrt{G_1G_4}-\frac{1}{2G_4}(G_3-\vartheta)^2+0(\mu_c^2). \tag{4.11}$$

As shown in [16], the set of states φ_n, χ_n is a biorthonormal one:

$$\int_{-\infty}^{+\infty}\chi_m(Z)\varphi_n(Z)\,\mathrm{d}Z=\delta_{mn} \tag{4.12}$$

provided that the normalization constant is chosen as

$$N_m=\bar{N}_m^*=\exp[(m+\tfrac{1}{2})\,\mathrm{tgh}^{-1}(\omega_2)] \tag{4.13}$$

with $\omega_2=1-\mu_c\sqrt{G_4/G_1}$.

Let us evaluate the nH Berry's phase for the FEL supermode system. In this case, the problem is one-dimensional, the parameter being now $Z=Z(\tau)$. A straightforward application of (2.15) then gives ($\nabla_R=\partial/\partial Z$):

$$\bar{\gamma}_n(C)=2\mathrm{i}\frac{L_c}{\mu_c}\frac{(G_3-\vartheta)}{G_4}=\bar{\gamma}(C) \tag{4.14}$$

independent of n.

Let us further clarify what it physically means, in the context of FEL operation, to perform a closed path in a one-dimensional parameter space. Firstly, since the time τ is a discrete dimensionless time, related to the cavity round-trip period, the quantity which is adiabatically changed is actually the delay parameter ϑ (connected to the cavity mismatch δL from the nominal round-trip condition: see table 1). Then, using as reference packet the electron bunch, the optical packet will perform, varying ϑ, a closed path around the maximum of the electron bunch distribution (see figure 2) or better a back and forth trip between the positions of maximum overlapping between the electron and laser bunches ($Z=-\Delta/2$ and $Z=\Delta/2$, Δ being the slippage).

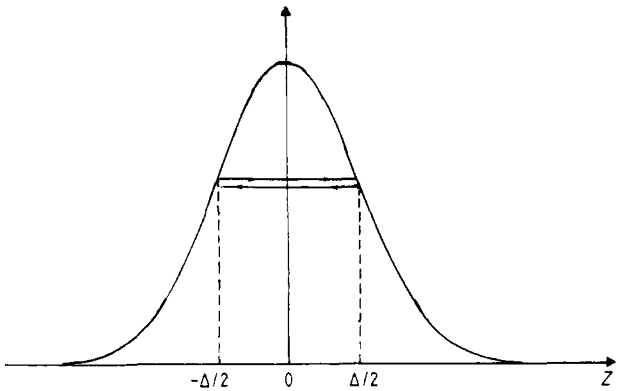

Figure 2. Schematic representation of the back and forth trip of the optical bunch between the positions of maximum overlapping with the electron bunch (full line).

Morever, it must be explicitly noticed that, in the FEL case, the existence of a topological phase is strictly related to the non-Hermicity of $\hat{H}$, which, in turn, is essentially linked to the operator

$$\Omega_3 \hat{a} = \mu_c (G_3 - \vartheta) \frac{d}{dZ} \tag{4.15}$$

i.e. the velocity term contribution to the FEL propagation equation.

It is immediately seen from (4.14) that the vanishing of the complex coefficient $G_3 - \vartheta$ of the velocity term (4.15) implies a null nH Berry's phase. Indeed, when

$$\operatorname{Re} G_3 = \vartheta \qquad \text{then} \qquad \operatorname{Im} G_3 = 0. \tag{4.16}$$

The FEL pseudo-Hamiltonian (4.2) is (almost) Hermitian† and the supermode system is no longer biorthogonal [16]. Let us recall that, from a physical point of view, the first of (4.16) is the condition to compensate for the lethargy effect due to the slowing of the radiation velocity [19].

Finally, it is to be stressed that the FEL provides an example of a *classical* system (obeying an nH Schrödinger-like equation), able to develop geometrical phase factors (see [20] for other examples of classical systems exhibiting a Berry's phase phenomenon).

5. Conclusions

We have exploited an approach based on the biorthonormal properties of the eigenstates of an NH operator and of its adjoint to study the evolution of non-conservative systems. We have shown that the wavefunction of a system ruled by an nH Hamiltonian and transported around a closed path in parameter space acquires, besides the standard, dynamical phase, a topological phase as well. This nH analogy of Berry's phase is, in general, complex, thus implying two damping factors (time dependent and path dependent) for the open-system wavefunction. Our result holds for both adiabatic and non-adiabatic evolution. In the latter case, the expression of the nH Berry's phase has been derived by a non-unitary evolution operator method.

† Actually, the gain functions G_α become almost real ($\operatorname{Im} G \sim 0$) at $\nu_0 = 2.6$ (where the FEL gain is maximum), see [19].

We have applied our results to the optical supermode propagation in the FEL, which is described, under suitable approximations, by an nH Schrödinger-like equation. The related Berry's phase is connected to the non-Hermiticity of the FEL Hamiltonian (i.e. to the so-called lethargic behaviour of the FEL).

Finally, let us briefly notice that the nH topological factor allows an interpretation analogous to that of the standard Berry's phase, i.e. it is nothing but the (complex) holonomy associated with the connection on a complex fibre bundle. These geometrical aspects of the nH Berry's phase will be discussed elsewhere.

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